PREDICTION OF SEISMIC WAVE AMPLITUDES USING THE PHASE-FRONT PARABOLIC APPROXIMATION

A. J. Haines*

SYNOPSIS

A computer package has been developed to predict the pattern of SH-wave amplitudes in the vicinity of an earthquake. The phase-front parabolic approximation, which can be applied where there are significant lateral variations in the elastic constants within the Earth, has been adapted so that it also allows for surface topography. Examples are presented from a study of shaking in the Hutt Valley and the surrounding region as a result of movement on the Wellington Fault.

INTRODUCTION

Methods for modelling the propagation of short-period seismic waves have been developed recently, requiring feasible amounts of computer time and storage, to estimate the amplitude of waves as they pass through geological structures that vary laterally as well as vertically. Previously, numerical solutions were only practical for problems where (i) the period of the waves is much longer than those periods that cause structural damage in the vicinity of an earthquake, or (ii) the velocity of the waves depends significantly on only the depth below the Earth's surface and there is effectively no surface topography. In the former case, solutions can be obtained by finite-differences (Boore 1972) and finite-elements (Smith 1975), and in the latter case, which is generally encountered only where there is no tectonic activity and is, therefore, of little interest when assessing seismic risk, solutions can be obtained using the Cagniard method (Helmberger 1968; Chapman 1976) and the reflectivity method (Kennett 1982).

The new methods are based on approximate, parabolic forms of the elastodynamic equation governing the propagation of seismic waves. Parabolic approximations have been used successfully in other fields (Tappert 1977). They have the advantage that far less computer time and storage are required to obtain numerical solutions than when the exact equations are solved. As well, parabolic approximations, unlike ray theory, allow for narrow-angle diffraction, which results in frequency-dependent smoothing of the disturbance as it propagates away from the source (Haines 1983b). They do not, however, compensate for energy lost when changes in the medium cause it to be scattered back towards the source, though by solving a sequence of parabolic equations, each of which allows for the back-scattered energy neglected by the previous equation in the sequence, it is possible to develop a complete picture of the wavefield (Tappert 1977; Haines 1983a).

Parabolic approximations are derived by, first, choosing a spatial coordinate such that the coordinate lines along which it varies are roughly parallel to the direction of propagation of the disturbance being

considered. The dependent variables in the equation of motion are then chosen so that their second derivatives with respect to this coordinate can be neglected, which is possible only if the wavelength of the disturbance is small compared with the scale-length of variations in the material properties of the medium. A number of parabolic approximations have been derived for seismic waves in situations where, as well as being slowly varying, the wave-speed is almost uniform and, in addition, the disturbance closely resembles plane waves (Landers and Claerbout 1972; McCoy 1977; Hudson 1980).

Two parabolic approximations for seismic waves have been derived for use under quite general conditions. They are the Gaussian beam approach (Cerveny, Popov and Psencik 1982; Cerveny and Psencik 1983a,b; Cerveny 1983) and the phase-front approach (Haines 1983b, 1984a,b). The Gaussian beam method is an extension of ray theory which involves solving separate parabolic equations for the P- and S-wave displacement fields in the vicinity of every ray emanating from the source. Its virtue is that one can compute the P- and S-wave displacements at isolated receiving points for very-high-frequency disturbances, which are such that once seismic energy starts propagating along a given ray then, neglecting back-scattering, the energy is always trapped close to that ray. The phase-front approach, on the other hand, is designed to give the P- and S-wave displacements over a broad region surrounding the source, and is applicable at lower frequencies. The method is based on the fact that as disturbances whose wavelengths are small, compared with the scale-lengths of the variations in material properties they encounter, move away from the source they do so in a fashion such that their directions of propagation are more or less perpendicular to the lines of peaks and troughs, or phase-fronts, in the displacement field (Haines 1983b, 1984a,b). This is what happens when a pebble is dropped into a pool of water. For each problem, a spatial coordinate is chosen so that the surfaces on which the coordinate is constant will roughly coincide with the lines of peaks and troughs. This coordinate varies in the direction away from the source. Separate equations for the P- and S-wave displacement fields are solved by successively determining the displacement field on each of a finite set of surfaces on which the coordinate is constant.

*Seismological Observatory, Geophysics Division, DSIR, Wellington, New Zealand.

In this paper a computer package based on the phase-front method is described, which is intended to help in micro-zoning. The package has been designed so that each programme will run on the small PDP 11/34 computer, with 60 Kbytes of accessible memory, that is operated by the New Zealand Seismological Observatory. Because of this restriction, it is only possible to consider 2-dimensional structures, and to model the SH-wave motion generated by a strike-slip source. The motion that is modelled is at points in the vertical plane through the centre of the source and perpendicular to the strike of the variations in material properties, which have the same strike as the fault. SH ground displacements are the horizontal displacements in the plane perpendicular to the direction from the point to the source, and, as is well known, they are the largest displacements that are observed as a result of strike-slip movement (in 2-dimensional media). In other words, the fact that we cannot consider the P and SV ground displacements is unimportant when assessing seismic risk due to such sources.

The package has features not included in previous implementations of the phase-front approach. One such feature is the facility to allow for topographic features that occur perpendicular to the strike of the fault. This is done by constructing a conformal mapping such that the Earth is transformed into a flat half-space, and solving the elastodynamic equation using the new coordinate system. As will be shown in a future paper, in this case the reflection of SH waves by the Earth's surface can be handled using an image technique originally developed for flat surfaces (Haines 1983b): the hypothetical medium through which the waves are assumed to propagate is taken to be the region below the Earth's surface welded to its mirror image above the Earth's surface, and at each point below the surface the actual displacement is the sum of two parts, the direct-wave displacement which we calculate at that point and the reflected-wave displacement which we calculate at the mirror image point above the surface.

The other significant innovation is first-order corrections for the geometrical spreading that arises when the phase-fronts are not cylindrical; that is, their shape changes along the strike of the fault. As a result, one can allow for the fact that the length of fault that slips during an earthquake, measured along the strike, is finite. This facility is important for small-to-moderate earthquakes where the length is of the same order or smaller than the distances from the source to points at which one would like to know the intensity of shaking. For larger earthquakes one can override the facility with the effect that the source is viewed as being infinite in extent.

INPUT TO THE PROGRAMMES AND THE METHOD OF SOLUTION

Topography

The computer programmes are designed to run in sequence. The first programme determines the conformal mapping that

transforms the Earth into a flat half-space. Several values of horizontal position x and the corresponding altitudes y , measured downwards from a reference surface, are read into the computer, and a smoothly-varying surface, consisting of points $(x(u), y(u))$ where u is variable, is constructed so that it closely matches the surface obtained by joining the points that are input. The functions $x(u)$ and $y(u)$ are taken to be the real and imaginary parts of a complex-valued function

$$z(u) = x(u) + iy(u);$$

in mathematical terminology, they are constrained to be a Hilbert-transform pair. This, together with the requirement that the smoothly-varying surface be as close as possible to the input surface, is sufficient to define the functions: one obtains what is referred to as a fixed point problem, which is solved iteratively. Once $z(u)$ is known as a function of the real-valued variable u , it can be generalised so that it is an analytic function of the complex-valued variable

$$w = u + iv.$$

The conformal mapping from z to w , or equivalently from (x,y) to (u,v) , transforms the Earth into a flat half-space, such that u is the coordinate along the surface of the half-space and v is the depth below the surface (Figure 1).

The Velocity Model

Secondly, one inputs the velocity model for SH waves in the region under consideration. In the main programme, which is discussed below, the corresponding wave-speeds in the transformed space, where u and v are the coordinates and the Earth is flat, are used instead of the actual velocities. These two quantities are related as follows:

$$\beta_w = \beta_z / \bar{h}$$

where β_z and β_w are the wave speeds in the (x,y) and (u,v) planes respectively and

$$\bar{h} = \left| \frac{dz}{dw} \right|$$

is the ratio of distances in the (x,y) plane to the corresponding distances in the (u,v) plane. At points corresponding to those below hills in the real Earth (the (x,y) plane), where $\bar{h}$ is large, the waves appear to travel slowly in the (u,v) plane: the converse is true below valleys. In non-mathematical terms, the wave speed is adjusted to allow for the difference in distance scales between the two spaces.

The Earthquake Source

Several parameters are used to define the earthquake source in the main programme. For the sake of simplicity, the source is taken to have a dipolar radiation pattern, corresponding to strike-slip movement concentrated near the centre of the source (Aki and Richards 1980). The location of this point in the (x,y) plane is specified, as is the dip of the fault. Other parameters are the frequency of the

Figure 1: Conformal mapping from the (x,y) plane to the (u,v) plane, showing a circle of radius $\tilde{r}_w$ in the (u,v) plane and the corresponding surface in the (x,y) plane. u and v are constant along the vertical and horizontal lines respectively.

disturbance, a distance R_s of approximately one wavelength from the source, inside which the solution is assumed to be of dipolar form, and the maximum amplitude of the disturbance at that distance; this amplitude occurs at the two points such that the line joining them to the centre of the source is perpendicular to the fault (Aki and Richards 1980). The description of the source is completed by specifying the radius of curvature R_1 , along the strike of the fault, of the phase-fronts at the distance R_s from the source. If

$$R_1 = R_s$$

the source is a point source in 3-dimensional space, whereas if

$$R_1 = \infty$$

the source is a line source:

$$R_1 = R_s$$

corresponds to a very small earthquake and

$$R_1 = \infty$$

corresponds to a very large earthquake.

Application of the Phase-Front Method

Unlike previous implementations of the phase-front method, the computer package models disturbances that are time-harmonic. A general requirement with the phase-front method is that the frequency content of the source must be concentrated in a narrow band - when this is not the case, the temporal variation of the source is decomposed into a sequence of narrow-frequency-band signals (Haines 1984a). This condition is satisfied trivially when there are only waves of one frequency. As usual, the advantage with time-harmonic disturbances is that their time-dependence is contained entirely within the exponential function

$$e^{-i\omega t}$$

where ω is the (angular) frequency. Consequently, one does not have to model their evolution with time, and so there is a considerable saving in computer storage. On the other hand, in its most accurate form, the approximate parabolic equation, used in the phase-front approach, involves

$$k_\xi = \frac{1}{h} \frac{\partial}{\partial \xi} \arg(u_3)$$

which is the wave-number in the direction away from the source; ξ is a curvilinear coordinate which varies away from the source in much the same fashion as the phase $\arg(u_3)$ of the SH-wave displacement u_3 , and

$$\frac{1}{h} = |\nabla \xi|$$

For time-harmonic waves, one can either solve the parabolic equation iteratively, using the previous value of k_ξ at each iteration, or replace k_ξ with some known function, which will generally be ω/β where β is the appropriate wave speed (β_s or β_w depending on whether one is working in the (x,y) or (u,v) plane. In the computer package, the latter approach is adopted.

The principal disadvantage in replacing k_ξ with ω/β is that, if one were to combine the solutions to the parabolic equation at different frequencies, to obtain a disturbance whose amplitude varies with time, one would find that the energy contained in the waves propagates with a slightly incorrect group velocity (Haines 1984a).

Another assumption made in the phase-front approach is that the coordinate ξ can be chosen so it is smoothly varying and the surfaces on which ξ is constant are roughly parallel to the phase-fronts in the displacement field. We take ξ to be the distance $\tilde{r}_w$ from the source to each point in the (u,v) plane (see Figure 1). It is possible to choose ξ so that one obtains better agreement between the surfaces of constant ξ and the phase-fronts than is obtained using a simple function like $\tilde{r}_w$ (Haines 1983b, 1984b). All the same, given that we have very little computer storage, $\tilde{r}_w$ is the most convenient function to use.

The approximate, parabolic equation we solve in the (u,v) plane has a simple form:

$$\frac{2i\omega}{\beta_w \sqrt{r_{1w} \beta_s r_{1w}}} \frac{\partial}{\partial \tilde{r}_w} (u_3 \sqrt{F_w \beta_s r_{1w}}) + \frac{2\omega^2}{\beta_w^2} u_3 + \frac{1}{\tilde{r}_w^2} \frac{\partial^2 u_3}{\partial \theta_w^2} = 0$$

where r_1 is the radius of curvature along the strike of the fault of the phase-front through each point in the (x,y) plane, and θ_w is the azimuth from the source of each point in the (u,v) plane. This equation will be derived in a future paper. There it will be shown, using Snell's Law, that to the first level of approximation r_1 is given by

$$\beta_s r_{1w} = \beta_s R_1 + \int_{\tilde{R}_w}^{\tilde{r}_w} \beta_s h d\tilde{r}_w$$

where $\tilde{R}_w = R_s/h_s$,

R_s and R_1 are as defined in the second section, and h and β_s are the values of h and β_s at the source. To solve the parabolic equation numerically one starts with the value of u_3 at each point on the circle with

$$\tilde{r}_w = \tilde{R}_w$$

- the values of u_3 on this circle are given by the dipolar radiation pattern of the source. Then one uses the parabolic equation to obtain the values of u_3 on the circle with

$$\tilde{r}_w = \tilde{R}_w + \delta \tilde{r}_w$$

where $\delta \tilde{r}_w$ is the increment in $\tilde{r}_w$. From the values of u_3 on the circle

$$\tilde{r}_w = \tilde{R}_w + \delta \tilde{r}_w$$

one obtains the values of u_3 on the next circle, and so on.

OUTPUT: HUTT VALLEY EXAMPLE

Two programmes plot the output from the other programmes. One plots the velocity model. An example is shown in

(UNTRANSFORMED) VELOCITY MODEL FOR GRENADA NORTH - WAINUIOMATA

Figure 2: S-wave velocity model for the 12 km wide region from Grenada North to Wainuiomata, which surrounds the Hutt Valley. Velocity contours are plotted every 0.25 km/s from 1.25 km/s to 3.25 km/s. The triangles are knot-points at which the velocity is specified so that the values of β_w can be interpolated in the main programme: certain of these points are chosen by the computer to ensure that variations in $\bar{h}$ are taken into account. The actual (untransformed) velocity β_z is 1.0 km/s at the Earth's surface, 1.5 km/s at the bottom of the gravels in the Hutt Valley, 2.5 km/s at the top of the basement rock underlying them, and 3.5 km/s at the bottom layer of knot-points, which is 5 km deep. There is no vertical exaggeration. The Wellington Fault is at the western margin of the gravels and coincides with the vertical line of knot-points in the centre of the diagram.

GRENADA NORTH - WAINUIOMATA RUN 1 WITH FREQ 4 HZ: REFLECTED PHASE

Figure 3: Surfaces of constant phase for the reflected wavefield resulting from 4 Hz strike-slip movement 3 km deep on the Wellington Fault. The lines of plusses (+) are separated from the lines of minusses (-) by half a cycle of the waves; that is, the plusses indicate the position of the peaks in the wavefield at a particular instant in time and the minusses indicate the positions of the troughs at that same instant.

GRENADA NORTH - WAINUIOMATA RUN 1 WITH FREQ 4 HZ: REFLECTED AMPLITUDE

Figure 4: Surfaces of constant amplitude for the reflected wavefield generated by the same source as in Figure 3. The relative amplitudes are plotted using the semi-logarithmic scale in Table 1 - in general, one specifies the amplitude corresponding to each of up to 10 symbols.

Figure 2. The model, taken from Cowan and Hatherton (1968) and Hochstein and Davey (1974), is for a cross-section of the region surrounding the Hutt Valley, and the Wellington Fault, which runs along the western side of the valley, is where the abrupt vertical change in velocity occurs in the centre of the cross-section. The function contoured is β_z ; as well, the programme will plot β_w , the transformed velocity that applies in the (u,v) plane.

The other programme plots the phase and amplitude of the direct and reflected waves. Figures 3 and 4 show the phase and amplitude of the reflected wavefield for a 4 Hz disturbance located on the Wellington Fault at the depth of 3 kilometres. The phase-fronts in Figure 3 are roughly perpendicular to the direction in which seismic energy propagates. There are several nodes, or places where the amplitude is zero and the phase changes by half a cycle. In Figure 4, the topography results in focusing and defocusing of the reflected energy. In general, for disturbances whose wavelengths are small compared with the wavelength of the topography, anticlinal topography acts like a concave mirror which focuses the reflected energy, whereas synclinal topography defocuses it, much as one would expect.

Another example is shown in Figures 5 and 6. A 1 Hz line source is placed at the Earth's surface, and the wavelength of the disturbance is comparable to the wavelength of the topography west of the fault. Here and in the previous example, the fault is assumed to be vertical. The amplitude pattern west of the fault (Figure 5) is characteristic of fundamental-mode Love waves, which are such that the amplitude is almost uniform near the Earth's surface and then decays away exponentially with depth. For a surface source, the direct and reflected wavefields are identical. East of the fault, the wave energy is trapped in the gravels below the Hutt Valley, and the second Love-wave mode forms, with a node near the interface between the gravels and the underlying basement (Figure 6). Though there is a sharp change in material properties across this interface, the main programme obtains the solution by, in effect, assuming that the wave speed always varies smoothly between the gridpoints at which the displacement is computed. Consequently, in general, the boundary conditions at such interfaces are not satisfied exactly.

INTERPRETING THE RESULTS

When deciding where the shaking is likely to be greatest during an earthquake at a given site, one has to take into account the area over which faulting occurs and the frequency content of the source. By introducing separate values for R_z and R_l we have allowed for the extent of the source along the strike of the fault: provided that R_l is not very much larger than R_z , the length of fault that slips is approximately equal to

$$R_l - R_z;$$

in the first example (Figures 3 and 4), R_z is 0.6 kilometres, R_l is 1.5 kilometres, and so the length of fault is approximately

0.9 kilometres. However, to allow for the extent of the source in the (x,y) plane and for its frequency content, one has to rerun the main programme with sources at different depths and with different frequencies. In the case of shallow earthquakes on the Wellington Fault, we found that the wave amplitudes in the Hutt Valley will be typically one and a half times as large as the amplitudes at the corresponding positions west of the fault. This is in general agreement with observations made by Smith and Maunder (1980), who interpret the level of micro-seismic noise near the Hutt Valley to mean that the intensity of shaking, during an earthquake in that vicinity, will be at least one unit (on the Modified Mercalli scale) greater in the Hutt Valley than elsewhere in the region (W D Smith personal communication).

ACKNOWLEDGEMENTS

Thanks are due to Dr W D Smith for criticism of the manuscript.

REFERENCES

1. Aki, K and Richards, P G (1980), "Quantitative seismology", two volumes, W H Freeman, San Francisco.
2. Boore, D M (1972), "Finite difference methods for seismic wave propagation in heterogeneous materials", in "Seismology: surface waves and Earth oscillations", Methods in computational physics 11, Academic Press, New York.
3. Cerveny, V, Popov, M M and Psencik, I (1982), "Computation of seismic wave fields in homogeneous media - Gaussian beam approach", Geophys. J R astr. Society, 70, pp 109-128.
4. Cerveny, V and Psencik, I (1983a), "Gaussian beams in two-dimensional elastic media", Geophys. J R astr. Society, 72, pp 419-435.
5. Cerveny, V and Psencik, I (1983b), "Gaussian beams and paraxial ray approximation in three-dimensional elastic inhomogeneous media", J Geophys., 53, pp 1-15.
6. Cerveny, V (1983), "Synthetic body wave seismograms for laterally varying layered structures by the Gaussian beam method", Geophys. J R astr. Society pp 389-426.
7. Chapman, C H (1976), "Exact and approximate generalised ray theory in vertically inhomogeneous media", Geophys. J R astr. Society, 46, pp 201-233.
8. Cowan, M and Hatherton, T (1968), "Gravity surveys in Wellington and Hutt Valley", NZ Journal Geol. Geophys., 11, pp 1-15.
9. Haines, A J (1983a), "Parabolic approximations for seismic disturbances in laterally varying structures", two volumes, unpublished PhD disser-

tation, University of Cambridge.

10. Haines, A J (1983b), "A phase-front method - I. Narrow-frequency-band SH-waves", *Geophys. J R astr. Society*, 72, pp 783-808.
11. Haines, A J (1984a), "A phase-front method - II. Broad frequency band SH-waves", *Geophys. J R astr. Society*, (in press).
12. Haines, A J (1984b), "A phase-front method - III. Acoustic waves, P- and S-waves", *Geophys. J R astr Society* (in press).
13. Helmberger, D V (1968), "The crust-mantle transition in the Bering Sea", *Bulletin Seism. Soc. Am.*, 55, pp 179-214.
14. Hochstein, M P and Davey, F J (1974), "Seismic measurements in Wellington Harbour", *Journal Royal Society of New Zealand*, 4, pp 123-140.
15. Hudson, J A (1980), "A parabolic approximation for elastic waves", *Wave Motion*, 2, pp 207-214.
16. Kennett, B L N (1982), "Seismic wave propagation in stratified media", Cambridge University Press.
17. Landers, T and Claerbout, J F (1972), "Numerical calculations of elastic waves in laterally inhomogeneous media", *J. Geophys. Res.*, 77, pp 1476-1482.
18. McCoy, J J (1977), "A parabolic theory of stress wave propagation through inhomogeneous linearly elastic solids", *Journal Appl. Mech.*, 44, pp 462-468.
19. Smith, W D (1975), "The application of finite element analysis to body wave propagation problems", *Geophys. J R astr. Society*, 42, pp 747-768.
20. Smith, W D and Maunder, R D (1980), "Microseismic noise survey in Lower Hutt", *Geophysics Division Report 142*, DSIR, Wellington.
21. Tappert, F D (1977), "The parabolic equation method", in "Wave propagation in underwater acoustics", *Lecture Notes in Physics 70*, Springer-Verlag, Berlin.

TABLE 1 AMPLITUDE SCALE USED IN FIGS 4, 5 AND 6

Symbol	1	2	3	4	5	6	7	8	...
Relative Amplitude	1	2	3	4	6	8	12	16	...

Figure 5: Surfaces of constant amplitude west of the Wellington Fault for a 1 Hz line source at the Earth's surface, again using the scale in Table 1.

Figure 6: Amplitudes east of the fault. Otherwise everything is the same as for Figure 5.